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Interval observer for SIR epidemic model subject to uncertain seasonality

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Abstract The issue of state estimation is considered for a SIR model with seasonal variations and uncertainties in the transmission rate. Assuming continuous measurement of the number of new infectives per unit time, a class of interval observers with estimate-dependent gain is constructed, and asymptotic error bounds are provided. Mathematics Subject Classification (2000) Primary 92D30; Secondary 34C12, 93B07, 93D09, 93D30

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1 Introduction, presentation of the SIR model

The SIR model with vital dynamics, see e.g. [3,10], is one of the most elementary compartmental models of epidemics. It describes the evolution of the relative proportions of three classes of a population of constant size, namely the susceptibles S, capable of contracting the disease and becoming infective; the infectives I, capable of transmitting the disease to susceptibles; and the recovered R, permanently immune after healing. This model is as follows:

$$\dot{S} = \mu - \mu S - \beta SI \tag{1a}$$
$$\dot{I} = \beta SI - (\mu + \gamma)I \tag{1b}$$

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The natural birth and mortality rate is μ (the disease is supposed not to induce supplementary death rate), γ is the recovery rate, while β represents the transmission rate per infective. All these parameters are positive. We consider here *proportions* of the population: $S + I + R \equiv 1$. Notice that the dynamics of R ($\dot{R} = \gamma I - \mu R$) is unnecessary, as the total population size is constant.

When the parameters are constant, the evolution depends closely upon the ratio $\mathcal{R}_0 := \frac{\beta}{\mu+\gamma}$ [3,10]. When $\mathcal{R}_0 < 1$, the disease-free equilibrium is the only equilibrium and it is globally asymptotically stable. It becomes unstable when $\mathcal{R}_0 > 1$, and an asymptotically stable endemic equilibrium then appears.

On the contrary, when the parameters are time-varying, complicated dynamics may occur [11]. We are interested here in estimating the population of the three populations, in an attempt to forecast future epidemic bursts. We use here techniques of interval observers, including output injection, in the spirit e.g. of [7,13,6]. The dynamics of the obtained error equation is seen as a linear uncertain time-varying positive system, whose asymptotic stability is ensured through the search of a common *linear* Lyapunov function and adequate choice of the gain as function of the state estimate.

The hypotheses on the model and some qualitative results are presented in Section 2. The considered class of observers is given in Section 3, with some a priori estimates and technical results. The main result is provided in Section 4, where the asymptotic error corresponding to certain gain choice is quantified. Last, illustrative numerical experiments are shown in Section 5.

2 Hypotheses on the model and preliminaries

We consider in the sequel that the transmission rate is subjected to uncertain seasonal variations. Relatively modest variations of this type are believed to have the capacity to induce large amplitude fluctuations in the observed disease incidence. This seems due to harmonic resonance, the seasonal forcing exciting frequencies close to the natural near-equilibrium oscillatory frequency [5].

One assumes that the transmission rate β is bounded by two functions $\beta_\pm$, available in real-time (all functions are supposed locally integrable):

$$\beta_-(t) \leq \beta(t) \leq \beta_+(t) \quad \text{for a.e. } t \geq 0 \tag{2}$$

(typically with $0 < \liminf_{t\to+\infty} \beta_-(t) \leq \limsup_{t\to+\infty} \beta_+(t) < +\infty$).

Our goal is to estimate lower and upper bounds for the three subpopulations. The unique available measurement is supposed to be the incidence rate $y = \beta SI$, i.e. the number of new infectives per time unit (accessible through epidemiological surveillance). With representation (1), y is not a state component, contrary e.g. to [2,4]. One sees easily that with this output, the system is detectable, but *not* observable at the disease-free equilibrium (where $I = 0$).

The following result provides qualitative estimates of its solutions.

Lemma 1 *Assume $S(0) \geq 0$, $I(0) \geq 0$ and $S(0) + I(0) \leq 1$. Then the same properties hold for any $t \geq 0$. The same is true with strict inequalities.*

Proof. Integrating (1a), (1b) yields $I(t) = I(0)e^{\int_0^t(\beta(\tau)S(\tau)-(\mu+\gamma)).d\tau}$, $S(t) = S(0)e^{-\int_0^t(\mu+\beta(\tau)I(\tau)).d\tau} + \mu\int_0^t e^{-\int_\tau^t(\mu+\beta I)}.d\tau$, which show that $S(t), I(t) \geq 0$ for any $t \geq 0$; while integrating the differential inequality $\dot{S}+\dot{I} \leq \mu(1-S-I)$ shows that $1-(S(t)+I(t)) \geq 0$ for any $t \geq 0$. The same formulas provide the demonstration in the strict inequality case. □

3 A class of nonlinear observer models

We consider in the sequel the following class of observers for system (1):

$$\dot{\hat{S}} = \mu - \mu\hat{S} - y + k_S(t)(y - \beta_S\hat{S}\hat{I}) \tag{3a}$$

$$\dot{\hat{I}} = y - (\mu+\gamma)\hat{I} + k_I(t)(y - \beta_I\hat{S}\hat{I}) \tag{3b}$$

where the time-varying gain components $k_S(t), k_I(t)$ are yet to be defined.

Lemma 2 *Suppose that for some $\varepsilon > 0$,*

$$k_S(t) \geq 1 \text{ whenever } \hat{S}(t) \leq \varepsilon, \qquad k_I(t) \geq -1 \text{ whenever } \hat{I}(t) \leq \varepsilon. \tag{4}$$

Assume $\hat{S}(0) \geq 0$, resp. $\hat{I}(0) \geq 0$. Then, for any $t \geq 0$, $\hat{S}(t) \geq 0$, resp. $\hat{I}(t) \geq 0$.

Proof. Verify directly that, under assumption (4), $\dot{\hat{S}} > 0$, resp. $\dot{\hat{I}} \geq 0$, in the neighborhood of a point where $\hat{S} = 0$, resp. $\hat{I} = 0$. This proves the result. □

Last, the following technical result will be needed.

Lemma 3 *Define $e_S := S - \hat{S}$, $e_I := I - \hat{I}$. Then,*

$$\begin{pmatrix} \dot{e}_S \\ -\dot{e}_I \end{pmatrix} = \begin{pmatrix} -(\mu + k_S\beta_S\hat{I}) & k_S\beta_S S \\ k_I\beta_I\hat{I} & -(\mu+\gamma+k_I\beta_I S) \end{pmatrix} \begin{pmatrix} e_S \\ -e_I \end{pmatrix} + SI \begin{pmatrix} k_S(\beta_S - \beta) \\ k_I(\beta - \beta_I) \end{pmatrix} \tag{5}$$

Proof. One has $\dot{e}_S = -\mu e_S + k_S(\beta_S\hat{S}\hat{I} - y)$ and $\dot{e}_I = -(\mu+\gamma)e_I + k_I(\beta_I\hat{S}\hat{I} - y)$. On the other hand, $\beta_S\hat{S}\hat{I} - y = (\beta_S - \beta)SI + \beta_S S(\hat{I} - I) + \beta_S\hat{I}(\hat{S} - S) = -SI(\beta - \beta_S) - \beta_S S e_I - \beta_S\hat{I}e_S$, and similarly for $\beta_I\hat{S}\hat{I} - y$. One deduces

$$\begin{pmatrix} \dot{e}_S \\ \dot{e}_I \end{pmatrix} = -\begin{pmatrix} \mu + k_S\beta_S\hat{I} & k_S\beta_S S \\ k_I\beta_I\hat{I} & k_I\beta_I S + \mu + \gamma \end{pmatrix} \begin{pmatrix} e_S \\ e_I \end{pmatrix} - SI \begin{pmatrix} k_S(\beta - \beta_S) \\ k_I(\beta - \beta_I) \end{pmatrix} \tag{6}$$

and finally (5) when using $-e_I$ instead of e_I. □

Notice that system (6) appears monotone for *nonpositive* gains, which is detrimental to its stability. This is not the case with system (5), which will now be used to construct interval observers.

4 Error estimates for interval observers

Notice first that system (1) is not evidently, or transformable into, a monotone system. The instances of (3) presented in the next result provide a class of interval observers with guaranteed speed of convergence.

Theorem 4 *Consider the two* independent *systems*

$$\dot{S}_+ = \mu - \mu S_+ - y + k_{S+}(t)(y - \beta_-(t)S_+I_-) \tag{7a}$$
$$\dot{I}_- = y - (\mu+\gamma)I_- + k_{I-}(t)(y - \beta_+(t)S_+I_-) \tag{7b}$$
$$\dot{S}_- = \mu - \mu S_- - y + k_{S-}(t)(y - \beta_+(t)S_-I_+) \tag{8a}$$
$$\dot{I}_+ = y - (\mu+\gamma)I_+ + k_{I+}(t)(y - \beta_-(t)S_-I_+) \tag{8b}$$

i. Assume that the gains are nonnegative functions of $S_\pm, I_\pm$, that fulfill (4) *for some $\varepsilon > 0$. If $0 \leq S_-(t) \leq S(t) \leq S_+(t)$ and $0 \leq I_-(t) \leq I(t) \leq I_+(t)$ for $t = 0$, then the same holds for any $t \geq 0$.*

ii. If in addition the gain components $k_{S\pm}(t), k_{I\mp}(t)$ are chosen such that

$$\beta_-(t)k_{S+}(t) - \rho_+\beta_+(t)k_{I-}(t) = \frac{\rho_+\gamma}{\rho_+I_-(t) + S_+(t)} \tag{9a}$$
$$\beta_+(t)k_{S-}(t) - \rho_-\beta_-(t)k_{I+}(t) = \frac{\rho_-\gamma}{\rho_-I_+(t) + S_+(t)} \tag{9b}$$

for fixed $\rho_\pm > 0$, then, writing $V_+(t) := (S_+(t) - S(t)) + \rho_+(I(t) - I_-(t))$, $V_-(t) := (S(t) - S_-(t)) + \rho_-(I_+(t) - I(t))$, the state functions $V_\pm$ are positive definite when the trajectories are initialized according to point i., and[1]

$$\forall t \geq 0,\ V_\pm(t) \leq \int_0^t e^{-\int_\tau^t \delta_\pm} \max\{k_{S\pm}(\tau), \rho_\pm k_{I\mp}(\tau)\} S(\tau)I(\tau)(\beta_+(\tau) - \beta_-(\tau))\, d\tau$$
$$+ e^{-\int_0^t \delta_\pm(\tau).d\tau} V_\pm(0), \qquad \textit{with } \delta_\pm(t) := \mu + \gamma\frac{\rho_\pm I_\mp(t)}{\rho_\pm I_\mp(t) + S_+(t)} \tag{10}$$

The proposed observers guarantee that the errors converge exponentially, with speeds $\delta_\pm(t)$ that smoothly vary from μ (in absence of infectives: $I_\pm(t) = 0$) to at most $\mu+\gamma$ (in case of outbreak, if $\rho_\pm I_\mp(t) \gg S_+(t)$). Recall that a linear time-invariant *monotone* system is asymptotically stable *iff* it admits a *linear* Lyapunov function of the type $V_\pm$ [1,9,12]. With this in mind, it may indeed be deduced from the proof (see in particular (11)) that the convergence speed of observer of type (7)-(8) is bound to be *at most equal to* $\mu+\gamma$ in presence of epidemics, and *cannot be larger than* μ in absence of infectives[2]. Notice that these bounds do not depend upon the values of $\beta_\pm$.

Last, observe that, with the estimate-dependent choice of the gain defined in (9), the error equations may be non monotone. However they fulfill the positivity and stability properties mentioned in the statement.

[1] In accordance with the usual convention, in the following formula the signs $\pm, \mp$ should be interpreted either everywhere with the upper symbols, or everywhere with the lower ones.

[2] We constrain the closed-loop system to be monotone, so not any closed-loop spectrum can be realized.

Proof. We show the results for system (7) only, system (8) is treated similarly.

• Introduce the error terms $e_{S+} := S_+ - S$ and $e_{I-} := I - I_-$. Applying Lemma 3 to system (7) with $\hat{S} = S_+, \hat{I} = I_-, k_S = k_{S+}, \beta_S = \beta_-, k_I = k_{I-}, \beta_I = \beta_+$ (and therefore $e_S = -e_{S+}, e_I = e_{I-}$) yields

$$
\begin{aligned}
&\begin{pmatrix} \dot{e}_{S+} \\ \dot{e}_{I-} \end{pmatrix} = -\begin{pmatrix} \dot{e}_S \\ -\dot{e}_I \end{pmatrix} \\
&= -\begin{pmatrix} -(\mu + k_{S+}\beta_- I_-) & k_{S+}\beta_- S \\ k_{I-}\beta_+ I_- & -(\mu+\gamma+k_{I-}\beta_+ S) \end{pmatrix} \begin{pmatrix} e_S \\ -e_I \end{pmatrix} - SI \begin{pmatrix} k_{S+}(\beta_- - \beta) \\ k_{I-}(\beta - \beta_+) \end{pmatrix} \\
&= \begin{pmatrix} -(\mu + k_{S+}\beta_- I_-) & k_{S+}\beta_- S \\ k_{I-}\beta_+ I_- & -(\mu+\gamma+k_{I-}\beta_+ S) \end{pmatrix} \begin{pmatrix} e_{S+} \\ e_{I-} \end{pmatrix} + SI \begin{pmatrix} k_{S+}(\beta - \beta_-) \\ k_{I-}(\beta_+ - \beta) \end{pmatrix} \\
&= \begin{pmatrix} -(\mu + k_{S+}\beta_- I) & k_{S+}\beta_- S_+ \\ k_{I-}\beta_+ I_- & -(\mu+\gamma+k_{I-}\beta_+ S) \end{pmatrix} \begin{pmatrix} e_{S+} \\ e_{I-} \end{pmatrix} + SI \begin{pmatrix} k_{S+}(\beta - \beta_-) \\ k_{I-}(\beta_+ - \beta) \end{pmatrix} \ .
\end{aligned}
$$

The off-diagonal terms of the Jacobian matrix are respectively $k_{S+}(t)\beta_-(t)S_+(t)$ and $k_{I-}(t)\beta_+(t)I_-(t)$, clearly nonnegative for a.e. $t \geq 0$ due to the hypotheses on the gain components (see Lemma 2). The corresponding system is therefore monotone [8,14], and any solution of (7) departing with $e_{S+}(0), e_{I-}(0) \geq 0$ verifies $e_{S+}(t), e_{I-}(t) \geq 0$ for any $t \geq 0$. This proves *i.*

• Writing $X := \begin{pmatrix} e_{S+} & e_{I-} \end{pmatrix}^\intercal$, notice that $V_+(X) := u^\intercal X$, for $u := \begin{pmatrix} 1 & \rho_+ \end{pmatrix}$, and V_+ and $\rho_+ > 0$ as in the statement. When X is initialized with nonnegative values, then this property is conserved (see point *i.*), so V_+ is positive definite and may be considered as a candidate Liapunov function.

Along the trajectories of (7), one has, using δ_+ defined in (10),

$$
\begin{aligned}
&\dot{V}_+(X) + \delta_+ V_+(X) \\
&= u^\intercal \begin{pmatrix} \delta_+ - (\mu + k_{S+}\beta_- I_-) & k_{S+}\beta_- S \\ k_{I-}\beta_+ I_- & \delta_+ - (\mu+\gamma+k_{I-}\beta_+ S) \end{pmatrix} X + SIu^\intercal \begin{pmatrix} k_{S+}(\beta - \beta_-) \\ k_{I-}(\beta_+ - \beta) \end{pmatrix} \\
&= \begin{pmatrix} \delta_+ - \mu + (\rho_+ k_{I-}\beta_+ - k_{S+}\beta_-)I_- & \rho_+(\delta_+ - \mu - \gamma) - (\rho_+ k_{I-}\beta_+ - k_{S+}\beta_-)S \end{pmatrix} X \\
&\quad + SI(k_{S+}(\beta - \beta_-) + \rho_+ k_{I-}(\beta_+ - \beta)) \ .
\end{aligned} \tag{11}
$$

Choosing the gain as in (9a) gives $\delta_+ - \mu + (\rho_+ k_{I-}\beta_+ - k_{S+}\beta_-)I_- = \frac{\gamma I_-}{I_- + \frac{S_+}{\rho_+}} - \frac{\gamma I_-}{I_- + \frac{S_+}{\rho_+}} = 0$ and $\rho_+(\delta_+ - \mu - \gamma) - (\rho_+ k_{I-}\beta_+ - k_{S+}\beta_-)S = -\frac{\gamma S_+}{I_- + \frac{S_+}{\rho_+}} + \frac{\gamma S}{I_- + \frac{S_+}{\rho_+}} \leq 0$. Formula (11) then yields $\dot{V}_+(X) + \delta_+ V_+(X) \leq SI(k_{S+}(\beta - \beta_-) + \rho_+ k_{I-}(\beta_+ - \beta)) \leq \max\{k_{S+}, \rho_+ k_{I-}\} SI(\beta_+ - \beta_-)$, which gives (10) by integration. This proves point *ii.* and achieves the proof. □

5 Numerical experiments

One takes $\mu = 0.02$/year, $\gamma = \frac{1}{20}$/day $= \frac{365}{20}$/year. The transmission rate $\beta(t)$ is taken as $\beta^*(1 + \varepsilon \cos(\omega t))$, with nominal value β^* such that $\mathcal{R}_0 = \frac{\beta^*}{\mu+\gamma} = 17$, $\varepsilon = 0.4$ and $\omega = 2.4$rad./year, close to the pulsation of the near-equilibrium natural oscillations. Their estimates are $\beta_\pm(t) = (1 \pm 0.8)\beta(t)$. Last, S and

I are initialized at 0.06 and 0.001, close to the perturbation-free equilibrium ($\varepsilon = 0$), and the observer initial conditions as 0 and 1 (lower and upper values). Figure 1 shows results for $\rho_\pm = 10^2, 10^3, 10^4$ and (compare with (9))

$$k_{I\mp}(t) = \frac{1}{\rho_\pm}\frac{\beta_\mp(t)}{\beta_\pm(t)}, \qquad k_{S\pm}(t) = 1 + \frac{1}{\beta_\mp(t)}\frac{\gamma}{I_\mp(t) + \frac{S_\pm(t)}{\rho_\pm}} .$$

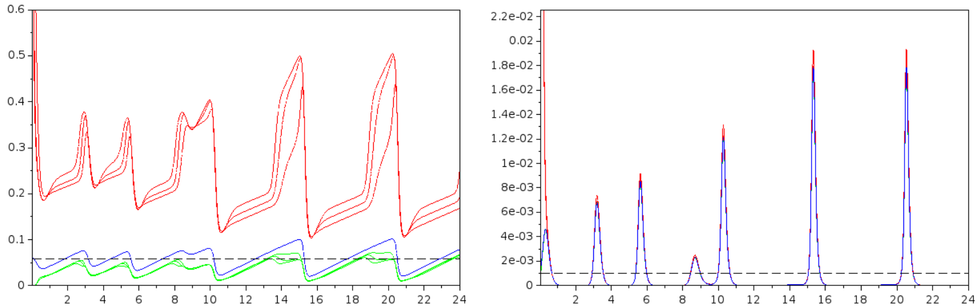

Fig. 1 Actual value (blue), lower estimates (green) and upper estimates (red) of S (left) and I (right) as functions of time (in years). The values at unperturbed equilibrium appear as dashed lines.

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